Solution for the fragment-size distribution in a crack-branching model of fragmentation

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(Received 2 March 2007; published 24 August 2007)

It is well established that rapidly propagating cracks in brittle material are unstable such that they generate side branches. It is also known that cracks are attracted by free surfaces, which means that they attract each other. This information is used here to formulate a generic model of fragmentation in which the small-size part of the fragment-size distribution results from merged crack branches in the damage zones along the paths of the propagating cracks. This model is solved under rather general assumptions for the fragment-size distribution. The model leads to a generic distribution $S^{-\gamma}\exp(-S/S_0)$ for fragment sizes S, where $\gamma=\frac{2d-1}{d}$ with d the Euclidean dimension, and S_0 is a material dependent parameter.

DOI: 10.1103/PhysRevE.76.026112 PACS number(s): 62.20.Mk, 46.50.+a, 61.43.Bn

INTRODUCTION

One of the early attempts to describe fragment-size distributions (FSDs) of explosive fragmentation goes back to the 1940s. Mott [1–3] experimented with fragmentation of thick shells and compared the results with those of a one-dimensional Poisson process and a two-dimensional random construction of horizontal and vertical lines dividing the plane into parts [4]. The latter model results in a cumulative FSD (i.e., the relative number of fragments with size larger than area S) of the form $N(S)\propto\sqrt{S}K_1(\sqrt{S})$, where K_1 is a modified Bessel function. This form of $N(S)$ is fairly close to the simple FSD of a one-dimensional Poisson process, $N(S)\propto\exp(-\sqrt{S})$ [5].

In the same spirit, Grady and Kipp [5] considered several different constructions of lines that divide a two-dimensional (2D) plane into parts. The ones that are most realistic as models of fragmentation do not allow lines to intersect. These typically result in FSD's close to that of a 2D Poisson process, $N(S)\propto\exp(-S)$ [notice the difference from the one-dimensional (1D) Poisson-process result above].

Another pair of classical papers on fragmentation are those of Gilvarry [6] and Gilvarry and Bergstrom [7]. Gilvarry derived an FSD under the assumption that uncorrelated flaws within the volume, on the surface, and along the edges of existing fragments are activated in an uncorrelated fashion. To fit the empirical Rossin-Rammler [8] and Gates-Gaudin-Schuhmann [9–11] FSDs, and the experimental FSD of Gilvarry and Bergstrom, he concluded that edge flaws dominate flaw activation in explosive fragmentation. This lead to an FSD for fragments of size between S and $S+dS$ in the form

$$n(S)\propto q(S)S^{-(d-1)/d}\exp(-S/S_0)dS, \qquad (1)$$

where d is the Euclidean dimension, S_0 relates to the frequency of the Poisson process, and $q(S)$ is the density of *a priori* fragments of size S. For the last parameter Gilvarry chose $q(S)=V_0/S$, where V_0 is the volume of the initial object.

Equation (1) provides excellent fits to numerous FSDs of fragmentation experiments, but there are some unresolved issues in the theory. In particular, it is not clear why the *a priori* density of fragments is $q(S)=V_0/S$. Furthermore, in the Gilvarry derivation, the cracks are assumed to form smooth crack surfaces. Relatively recently it has been established that rapidly propagating cracks (which is certainly the case in explosive fragmentation) in brittle materials are not stable, but crack branching and crack-tip splitting begin to appear beyond a critical crack velocity [12]. Such instabilities will destroy the smooth crack surface and small fragments will be formed in damage zones along the paths of the cracks. These small fragments will significantly affect the FSD in the small fragment-size limit [13], which makes the validity of the Gilvarry theory doubtful.

More recently it has been proposed that the distribution of distances between initiated crack branches may explain the shape of FSDs in some cases. This distribution has been found to have a log-normal shape [14,15]. An important ingredient missing in such a model is that cracks are attracted by a free surface (e.g., a crack surface left behind by another crack). A crack propagating beside a free surface (e.g., another crack) will turn toward this, and eventually merge with it. The crack cannot penetrate beyond the free surface and therefore it terminates there. As more of the crack branches merge, a decreasing number of remaining crack branches will form increasingly larger fragments as they propagate further away from the mother crack, expanding the damage zone around the crack. This process will continue until all crack branches have merged or stopped as a result of stress relaxation.

A numerical model based on this scenario was investigated by Inaoka and Takayasu [16,17]. Their numerical FSD's were consistent with the Gilvarry result,

$$n(S)\propto S^{-\gamma}\exp(-S/S_0)dS \qquad (2)$$

with $\gamma=\frac{3}{2}$ and $\gamma=\frac{5}{3}$ in 2D and 3D, respectively. These results indicate that $\gamma=\frac{2d-1}{d}$. In the cumulative FSD, $N(S)$, the power-law exponent is thus $\gamma-1=\frac{d-1}{d}$.

A number of fairly recent experiments have verified that this exponent γ describes the FSDs for a class of fragmentation processes in 2D [18] and three dimensions (3D) [19–21]. A recent review on the topic is in Ref. [22]. A direct experimental verification of fragment formation through

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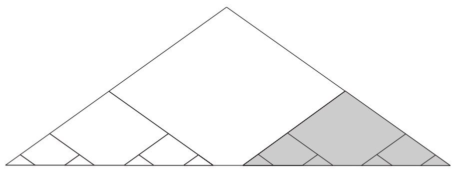

FIG. 1. 2D crack-branching model with triangular fragments. A third generation block is shown shaded.

crack-tip branching and merging of branches using a high-speed camera, is given in Ref. [23]. The mechanism of merging crack branches is also consistent with the results of a recently conducted experiment by Svahn [24]. In this experiment concrete blocks were exploded such that three concentric layers around the central blasting hole of the blocks were colored differently. In this way the spatial origin of fragments within the initial block could be identified with some accuracy. One-half of the small fragments were formed in the first layer around the hole and the other one-half in the two outer layers. The outermost layers of the blocks were cleaved into a few large fragments with small fragments produced around the cleaving cracks. This behavior strongly supports the formation of small fragments as a result of merging crack branches, as opposed, e.g., to having them formed next to the blasting hole only.

In this paper we use the experimental and numerical evidence described above, and construct a model for the production of small fragments in impact (or explosive) fragmentation of brittle material, based on merging of crack branches (daughter cracks) around an initial propagating crack. Under rather general assumptions concerning distances between branches, shape of the fragments formed, and probability of the cracks to get arrested in the material, we theoretically derive the resulting FSDs.

In the model introduced, angles between mother and daughter cracks, and between merging cracks, were not fixed, although tip splitting [12] and merger [25] typically involve characteristic angles. In fact we demonstrate that these angles do not affect the FSD. Notice that the merger angles are in reality close to 90°, but this angle results from a crack turning toward a neighboring free surface before merger.

TOY MODEL IN TWO DIMENSIONS

Before more general cases are considered, we first investigate three simple examples of our model in which the shape of the fragments is fixed. In the first case let the elementary fragments, formed by the first side-branch-merger generation closest to the mother crack, be isosceles triangles with base Δx and base angle α (Fig. 1). Triangular blocks defined by the baseline crack and a point of merger of two daughter cracks are similar in all generations, and the size of a block in generation k is given by

$$B_k = \frac{1}{4}(2^k - 1)(\Delta x)^2 \tan \alpha. \quad (3)$$

We can easily deduce from this result that the size of the new fragment generated in a kth generation merger is

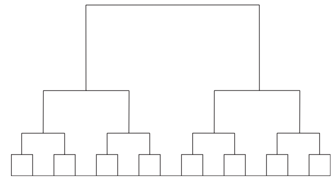

FIG. 2. 2D crack-branching pattern of squares.

$$A_k = \frac{1}{8}(4^k - 2)(\Delta x)^2 \tan \alpha. \quad (4)$$

In the second example let the elementary fragments be squares with edge Δx (Fig. 2). For the block sizes we find the recursion relation (k is a generation index)

$$B_1 = (\Delta x)^2, \quad B_{k+1} = 4B_k + (\Delta x)^2. \quad (5)$$

The solution of this difference equation is

$$B_k = \frac{1}{3}(4^k - 1)(\Delta x)^2, \quad (6)$$

and the size of the fragments in generation k is now

$$A_k = \frac{1}{6}(4^k + 2)(\Delta x)^2. \quad (7)$$

In our third simple example the elementary fragments are semicircles of diameter Δx (Fig. 3). In this case the block sizes satisfy the recursion relation

$$B_1 = \frac{\pi}{8}(\Delta x)^2,$$

$$B_{k+1} = 4B_k + \left(2^{k-1} + \frac{\pi}{8} - \frac{1}{2}\right)(\Delta x)^2. \quad (8)$$

This difference equation can be easily solved and we find

$$A_k = \left(\frac{2+\pi}{48}4^k + \frac{2\pi - 8}{48}\right)(\Delta x)^2 \quad (9)$$

for the size of the kth-generation fragments.

If the number of the elementary crack branches is 2^N, then in all these three cases the number of the kth generation fragments is $N_k = 2^{N-k}$, and for large k we get a power law

$$N_k \propto (A_k)^{-1/2}. \quad (10)$$

This power law follows from the fact that fragments are essentially similar in different generations, and the linear dimensions of the fragments are doubled in each step while their number is halved.

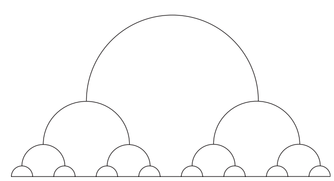

FIG. 3. 2D crack-branching model with semicircular fragments.

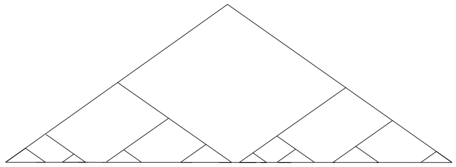

FIG. 4. 2D crack-branching model with triangular fragments of varying size.

All branches do not necessarily form fragments, and may stop spontaneously before reaching a free surface. Assume that in the first generation each fragment is formed with probability p. In the kth generation, assume that a fragment is formed with the same probability p, if all the possible fragments in the preceding generations of the same block exist. Now the number of fragments formed in generation k is binomially distributed,

$$N_k \sim \mathrm{bin}(p^{2^k-1}, 2^{N-k}), \tag{11}$$

and the expectation value of N_k obeys a power law with exponential cutoff,

$$\langle N_k \rangle \propto (A_k)^{-1/2} \exp\left(\frac{C}{\Delta x} (A_k)^{1/2} \ln p \right), \tag{12}$$

where constant C depends on the shape of fragments.

GENERALIZATION OF THE TOY MODEL

Consider again the triangle-fragment model, but now in the case where the fragments of any given generation can have different sizes (Fig. 4). Denote by Δx_i, $i=1,\ldots,2^N-1$, distances between the first-generation (elementary) daughter cracks. The block sizes in the kth generation are

$$B_k^j = \frac{1}{4}\left(\sum_{i=(j-1)2^k+1}^{j2^k-1} \Delta x_i \right)^2 \tan\alpha, \quad j=1,\ldots,2^{N-k}. \tag{13}$$

The fragment sizes in generation k satisfy the recursion relation

$$A_1^j = B_1^j, \quad j=1,\ldots,2^{N-1},$$

$$A_{k+1}^j = B_{k+1}^j - (B_k^{2j-1} + B_k^{2j}), \quad j=1,\ldots,2^{N-k-1}. \tag{14}$$

Assume now that distances Δx_i between elementary daughter cracks are independent and identically distributed with the expectation value and variance

$$\langle \Delta x_i \rangle = \Delta x, \quad \Delta^2(\Delta x_i) = \sigma^2. \tag{15}$$

For the expectation values of block and fragment sizes, respectively, we now find

$$B_k \coloneqq \langle B_k^j \rangle = \frac{1}{4}\left\langle \sum \Delta x_i \right\rangle^2 \tan\alpha$$

$$= \frac{1}{4}\{(\Delta x)^2 4^k + [\sigma^2 - 2(\Delta x)^2]2^k + (\Delta x)^2 - \sigma^2\} \tan\alpha \tag{16}$$

and

$$A_k \coloneqq \langle A_k^j \rangle = B_k - 2B_{k-1} = \frac{1}{8}\tan\alpha[(\Delta x)^2 4^k + \sigma^2 - 2(\Delta x)^2]. \tag{17}$$

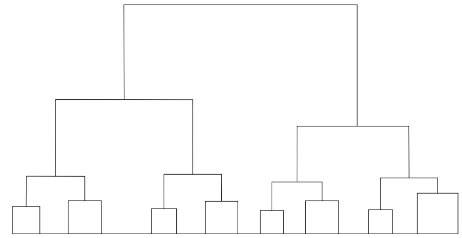

FIG. 5. 2D crack-branching model with square fragments of varying size.

We can modify the square-fragment model in an analogous way (Fig. 5). In order to have well-defined fragment generations, we must assume that the horizontal distance between the midpoints of the top edges of neighboring fragments is larger than the height of these fragments. This condition holds if we assume that $a < \Delta x_i < 3a$ for some $a > 0$. In this case the block sizes satisfy the recursion relation

$$B_1^j = (\Delta x_j)^2, \quad j=1,\ldots,2^{N-1},$$

$$4^k\left(B_{k+1}^j - \frac{1}{2}B_k^{2j+1} - \frac{1}{2}B_k^{2j-1} \right) = \left(\sum_{i=1}^{2^k} i\Delta x_{i+(j-1)2^{k+1}} + \sum_{i=1}^{2^k-1} i\Delta x_{j2^{k+1}-i} \right)^2,$$

$$j=1,\ldots,2^{N-k-1}. \tag{18}$$

Assume again that the Δx_i's are independent and identically distributed such that $a < \Delta x_i < 3a$, and

$$\langle \Delta x_i \rangle = \Delta x, \quad \Delta^2(\Delta x_i) = \sigma^2.$$

For the expectation values of the block sizes, $B_k = \langle B_k^j \rangle$, we obtain

$$B_1 = (\Delta x)^2 + \sigma^2,$$

$$4^k(B_{k+1} - B_k) = (\Delta x)^2 + \frac{4^{-k}}{3}(2^{k+1} + 2^{-k})\sigma^2. \tag{19}$$

In the same way as in the previous case we find

$$A_k = \frac{1}{6}(\Delta x)^2 4^k + \frac{1}{3}(\Delta x)^2 + 2^{1-k}\sigma^2 \tag{20}$$

for the expectation values of the fragment sizes, $A_k = \langle A_k^j \rangle$. In both these generalized cases we again obtain a power-law dependence between the number and average size of fragments in generation k,

$$N_k \propto (A_k)^{-1/2}.$$

Furthermore, if each fragment in the first generation is formed with probability p, and in the kth generation a fragment is formed with the same probability p if all fragments of the preceding generations in the same block exist, then the expectation value of N_k obeys a power law with an exponential cutoff,

$$\langle N_k \rangle \propto (A_k)^{-1/2} \exp\left(\frac{C}{\Delta x} (A_k)^{1/2} \ln p \right),$$

where constant C depends on the shape of fragments.

We can also obtain the power-law expression for $\langle N_k \rangle$ in a more general case. Assume that blocks are essentially similar, i.e., the area of a block can be written in the form

$$B_k^j = \alpha (l_k^j)^2 + \delta_k^j, \tag{21}$$

where l_k^j is a (weighted) sum of the Δx_i's that belong to block B_k^j, α is a constant, and δ_k^j is a small correction that depends on the shape of the block. If the Δx_i's are independent and identically distributed with

$$\langle \Delta x_i \rangle = \Delta x, \quad \Delta^2(\Delta x_i) = \sigma^2,$$

then the expectation value of the block size can be expressed in the form

$$B_k = \alpha (\Delta x)^2 4^k + \beta_k, \tag{22}$$

where β_k depends on Δx and σ and possibly also on higher moments of Δx_i. Now the expectation value of the fragment size is given by

$$A_k = \frac{1}{2} \alpha (\Delta x)^2 4^k + (\beta_k - \beta_{k-1}). \tag{23}$$

If the second term on the right-hand side of this equation is small compared to the first one, we get a similar power-law distribution as before. This generalization allows for fragments of varying size and varying shape.

CONTINUUM LIMIT

Consider next the toy models in the limits $\Delta x \to 0$ and infinite size of the system. Let the length of the system be $L = N\ell$ and $\Delta x = L/2^N$. Consider the total area per unit length of fragments with size less than A,

$$\frac{1}{L} \sum_{k:A_k \le A} N_k A_k. \tag{24}$$

The size of fragments in generation k was [cf. Eqs. (4) and (9)]

$$A_k = c(\Delta x)^2 4^k + \alpha_k (\Delta x)^2, \tag{25}$$

where $\alpha_k \ll 4^k$ for large k. If we approximate this expression by letting $\frac{\alpha_k}{4^k} \to 0$, we find that $A_k \le A$ when

$$k \le \log_4 \left(\frac{A}{c\ell^2} \right) + N - \log_4 N^2 =: k_A. \tag{26}$$

Now we can compute the sum in Eq. (24),

$$\begin{aligned} \frac{1}{L} \sum_{k:A_k \le A} N_k A_k &= \frac{1}{N\ell} \sum_{k=1}^{k_A} c\ell^2 2^{N-k} \frac{N^2}{4^N} \left(4^k + \frac{\alpha_k}{c} \right) \\ &= c\,\ell\, N \sum_{k=1}^{k_A} 2^{k-N} + \ell \frac{N}{2^N} \sum_{k=1}^{k_A} \alpha_k 2^{-k} \\ &= 2c\,\ell \left(\sqrt{\frac{A}{c\ell^2}} - \frac{N}{2^N} \right) + \ell \frac{N}{2^N} \sum_{k=1}^{k_A} \alpha_k 2^{-k}. \end{aligned} \tag{27}$$

If $\frac{\alpha_k}{4^k} \to 0$ such that

$$\sum_{k=1}^{N} \alpha_k 2^{-k} = o\left(\frac{2^N}{N} \right), \tag{28}$$

we find in the limit $N \to \infty$ that

$$\frac{1}{L} \sum_{k:A_k \le A} N_k A_k \to 2c\,\ell \sqrt{\frac{A}{c\ell^2}}. \tag{29}$$

On the other hand, in this limit the sum equals

$$\int_0^A A dN(A), \tag{30}$$

where $N(A)$ is the number of fragments (per unit length). This means that we obtain a power-law expression for the density of fragments,

$$n(A) = \frac{dN(A)}{dA} = \frac{1}{c\ell^3} \left(\frac{A}{c\ell^2} \right)^{-3/2}. \tag{31}$$

If we assume that cracks propagate with a constant speed, then the area production rate of the fragments is constant.

Consider the case when fragments are formed with probability p as described above. In the limit $N \to \infty$ we must also scale the probability p, and for this we use the scaling

$$p \to p_N = p^{\Delta x/\ell} = p^{N/2^N}. \tag{32}$$

Consider now the expectation value of total area per unit length of fragments with size less than A,

$$\begin{aligned} \frac{1}{L} \left\langle \sum_{k:A_k \le A} N_k A_k \right\rangle &= \frac{1}{N\ell} \sum_{k:A_k \le A} \langle N_k \rangle A_k \\ &= \frac{1}{N\ell} \sum_{k=1}^{k_A} 2^{N-k} p_N^{2^k - 1} \left(c\ell^2 \frac{N^2}{4^N} (4^k + \alpha_k) \right) \\ &= \frac{c\,\ell}{p_N} \sum_{k=1}^{k_A} 2^{k-N} N p^{N 2^{k-N}} + \frac{N\ell}{2^N} \sum_{k=1}^{k_A} \alpha_k 2^{-k} p_N^{2^k - 1}. \end{aligned} \tag{33}$$

The last sum on the right-hand side vanishes in the limit $N \to \infty$ if $\frac{\alpha_k}{4^k} \to 0$ fast enough. The first sum on the right-hand side can be written as a Riemannian lower sum, and we get an expression

$$\sum_{k=1}^{k_A} 2^{k-N}Np^{N2^{k-N}} = 2\sum_{k=1}^{k_A} (2^{k-N}N - 2^{k-1-N}N)p^{N2^{k-N}} \le 2\int_0^{\sqrt{A/c\ell^2}} p^x dx = \frac{1}{c\ell^2}\int_0^A \left(\frac{y}{c\ell^2}\right)^{-1/2} \exp\left(\ln p\sqrt{\frac{y}{c\ell^2}}\right)dy. \quad (34)$$

In the same way we get a lower bound for the same sum in the form

$$\sum_{k=1}^{k_A} 2^{k-N}Np^{N2^{k-N}} \ge \frac{1}{2c\ell^2}\int_0^A \left(\frac{y}{c\ell^2}\right)^{-1/2} \exp\left(\ln p\sqrt{\frac{y}{c\ell^2}}\right)dy. \quad (35)$$

In the limit $N\to\infty$ we thus find that

$$\frac{1}{L}\left\langle \sum_{k:A_k\le A} N_k A_k \right\rangle \sim \frac{1}{\ell}\int_0^A \left(\frac{y}{c\ell^2}\right)^{-1/2} \exp\left(\ln p\sqrt{\frac{y}{c\ell^2}}\right)dy, \quad (36)$$

which means that the density of fragments is a power law with exponential cutoff,

$$n(A) = \frac{dN(A)}{dA} \sim \frac{1}{\ell^3}\left(\frac{A}{c\ell^2}\right)^{-3/2} \exp\left(\ln p\sqrt{\frac{A}{c\ell^2}}\right). \quad (37)$$

The area production rate slows down as $\exp[C(\ln p)t]$, assuming that cracks propagate with a constant speed. The total area of the fragments is proportional to $-1/\ln p$.

We can generalize this result to the case where distances between the first-generation side branches vary. The expectation value of fragment size is of the same form as that in the toy models,

$$A_k = c(\Delta x)^2 4^k + \alpha_k, \quad (38)$$

where α_k depends on Δx and σ and possibly on higher moments of Δx_i, and is small compared to the first term. If the number of first-generation fragments is large and the distribution of Δx_i is narrow, then distributions of fragments in different generations do not overlap much, and we can approximate the expectation value of the total area per unit length of fragments with size less than A in the same way as before by summing over the generations with expected size $A_k \le A$,

$$\left\langle \frac{1}{L}\sum_{k,j:A_k^j\le A} A_k^j \right\rangle \approx \frac{1}{\langle L\rangle}\sum_{k:A_k\le A} \langle N_k\rangle A_k. \quad (39)$$

If we now take the limit $N\to\infty$ such that $\langle\Delta x_i\rangle = \frac{N}{2^N}\ell$, we get a result similar to that of Eq. (37) for the density of fragments.

THREE-DIMENSIONAL CASES

In analogy with the toy model of triangular fragments, we can construct a three-dimensional model with pyramid (elementary) fragments of a square base. The idea of the model is shown in Fig. 6, where blocks of three consecutive generations are drawn. Blocks are similar pyramids and the volume of a block in the kth generation is given by

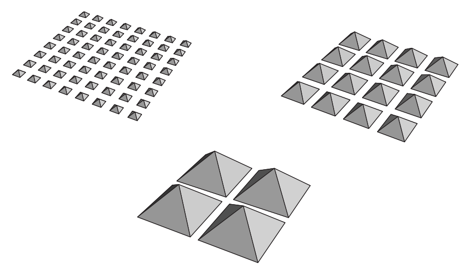

FIG. 6. 3D model of crack-branching fragmentation with pyramid-shaped elementary fragments. Blocks in three consecutive generations are shown.

$$B_k = \frac{1}{6}[(2^k-1)\Delta x]^3 \tan\alpha, \quad (40)$$

where α is the angle between the base and a side of the block, and Δx is the length of a base edge of the first-generation fragments. For the volume of fragments in generation k we find

$$V_k = \frac{1}{12}8^k\left[1 - \left(\frac{1}{4}\right)^k + 3\left(\frac{1}{8}\right)^k\right]\tan\alpha. \quad (41)$$

The model of square fragments can also be generalized into three dimensions with cubic elementary fragments. Figure 7 shows blocks in three consecutive generations. For block sizes we get the recursion relation

$$B_1 = (\Delta x)^3, \quad B_{k+1} = 8B_k + (2^{k+1}-1)(\Delta x)^3, \quad (42)$$

from which we find that

$$B_k = \left(\frac{4}{21}8^k - \frac{1}{3}2^k - \frac{1}{7}\right)(\Delta x)^3, \quad (43)$$

and further for the volume of fragments in generation k,

$$V_k = \left(\frac{2}{21}8^k + \frac{1}{3}2^k - 9\right)(\Delta x)^3. \quad (44)$$

If the number of elementary cracks is 2^N in both directions, then the number of fragments in generation k is

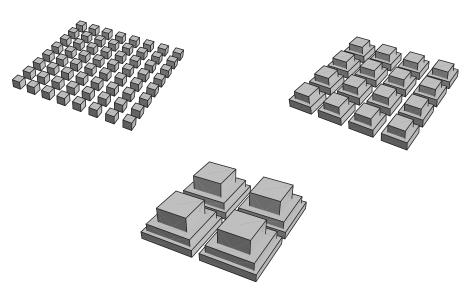

FIG. 7. 3D model of crack-branching fragmentation with cubic elementary fragments. Blocks in three consecutive generations are shown.

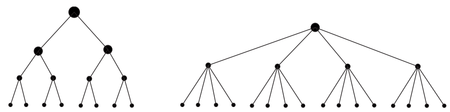

FIG. 8. Cayley-tree representation of the 2D and 3D models of crack-branching fragmentation.

$$N_k = (2^{N-k})^2, \tag{45}$$

which means that

$$N_k \propto (V_k)^{-2/3}. \tag{46}$$

We can now take a continuum limit in the same way as in the two-dimensional case above, and find thereby for the density of fragments

$$n(V) = \frac{dN(V)}{dV} \sim \frac{1}{\ell^5}\left(\frac{V}{c\ell^3}\right)^{-5/3}. \tag{47}$$

If we again assume that each fragment in the first generation is formed with probability p, and in the kth generation a fragment is formed with the same probability p if all the possible fragments of lower generations in the same block exist, then the number of fragments is binomially distributed,

$$N_k \sim \mathrm{bin}(p^{(4^k-1)/3}, 4^{N-k}). \tag{48}$$

For the expectation value of the number of fragments we find again a power law with exponential cutoff,

$$\langle N_k\rangle \propto (V_k)^{-2/3}\exp\left(\frac{C}{(\Delta x)^2}(V_k)^{2/3}\ln p\right), \tag{49}$$

where the constant C depends on the shape of the fragments. If we take a continuum limit with the scaling $p \to p_N = p^{(N/2^N)^2}$, we find

$$n(V) = \frac{dN(V)}{dV} \sim \frac{1}{\ell^5}\left(\frac{V}{c\ell^3}\right)^{-5/3}\exp\left[\frac{1}{3}\ln p\left(\frac{V}{c\ell^3}\right)^{2/3}\right] \tag{50}$$

for the density of fragments.

GENERIC MODEL OF CRACK-BRANCHING FRAGMENTATION

The models we introduced above can be expressed as regular Cayley trees as shown in Fig. 8. Vertices correspond to fragments and links describe how they form blocks. In the two-dimensional case every vertex in the kth generation has two parents and in the three-dimensional case there are four parents. The fragment sizes satisfy recursion relations of the form (generation number increases with increasing distance from the bottom line in Fig. 8)

$$\frac{A_{k+1}}{A_k} = 4 + \epsilon_k, \quad \frac{V_{k+1}}{V_k} = 8 + \epsilon_k, \tag{51}$$

where ϵ_k is a small correction.

Consider a Cayley tree in which every vertex in the kth generation has n daughters. If the vertex sizes satisfy the recursion relation

$$\frac{S_{k+1}}{S_k} = \alpha + \epsilon_k \tag{52}$$

with ϵ_k's small enough, then

$$S_k \sim C\alpha^k \tag{53}$$

for large k. If the number of vertices in the first generation is n^N, then the number of vertices in the kth generation is $N_k = n^{N-k+1}$, and we find that

$$N_k \propto (S_k)^{-\gamma}, \quad \gamma = \frac{\ln n}{\ln \alpha}. \tag{54}$$

Assume now that we remove a vertex in the first generation with probability p, and in the kth generation we remove a vertex with probability p if all the vertices in the same Caley-tree branch below it have been removed. The number of vertices removed in generation k is binomially distributed,

$$N_k \sim \mathrm{bin}(p^{(n^k-1)/(n-1)}, n^{N-k+1}), \tag{55}$$

and the expectation value of the number of fragments is a power law with exponential cutoff,

$$\langle N_k\rangle \propto (S_k)^{-\gamma}\exp[c(S_k)^{\gamma}\ln p]. \tag{56}$$

In dimensions $d=2$ and $d=3$ we find the familiar scaling exponents $\gamma = \frac{d-1}{d}$. Within the Caley-tree model we can generalize the system into d dimensions in an analogous way. If the fragments are essentially similar and the linear dimension of fragments is doubled in each generation, the fragment sizes satisfy the recursion relation

$$\frac{S_{k+1}}{S_k} = 2^d + \epsilon_k. \tag{57}$$

We thus find that $\alpha = 2^d$ in the Caley-tree model [cf. Eq. (52)]. The base of the fragmentation pattern is a $(d-1)$-dimensional square which is divided into subsquares by $2^{N(d-1)}$ elementary cracks. In each direction two elementary cracks meet to form secondary cracks in all generations. For the number of fragments we find $N_k/N_{k+1} = 2^{d-1}$. This means that $n = 2^{d-1}$, and we find in d dimensions that

$$N_k \propto (S_k)^{-\gamma} \tag{58}$$

and

$$\langle N_k\rangle \propto (S_k)^{-\gamma}\exp[c(S_k)^{\gamma}\ln p] \tag{59}$$

with $\gamma = \frac{d-1}{d}$. If we take a continuum limit in the same way as above, we find for the density of fragments

$$n(S) \sim \frac{1}{\ell^{2d-1}}\left(\frac{S}{\ell^d}\right)^{-(2d-1)/d} \tag{60}$$

and

$$n(S) \sim \frac{1}{\ell^{2d-1}}\left(\frac{S}{\ell^d}\right)^{-(2d-1)/d} \exp\left[C\left(\frac{S}{\ell^d}\right)^{(d-1)/d} \ln p\right], \tag{61}$$

if we allow stopping of side branches with probability p.

SUMMARY AND DISCUSSION

We have demonstrated that a generic geometrical model of fragmentation by merging side branches has a solution that is consistent with the FSD of the classical Gilvarry model and the FSD's typically found in blasting experiments. The model is inherently scale invariant and the resulting FSD has a scaling exponent $\gamma = -\frac{2d-1}{d}$. For large fragments the FSD may have an exponential cutoff, which results from spontaneous stopping of side branches (i.e., fragments are formed with a constant probability p).

It is important to notice that the model presented here only considers the small-fragment region of the FSD. It is possible that also the main cracks (i.e., the mother cracks of the side branches) merge with each other, and form thereby the large-fragment part of the FSD. For more or less uncorrelated main cracks, one would expect that FSD has in the large-fragment limit an (almost) exponential form. The exact functional form of FSD may thus depend on, e.g., the loading conditions of the object, and the large-size end of FSD is therefore not as generic as its small-size end.

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